

KLEIN-GORDON EQUATION FROM MAXWELL-LORENTZ DYNAMICS

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ABSTRACT. We consider Maxwell-Lorentz dynamics: that is to say, Newton's law under the action of a Lorentz's force which obeys the Maxwell equations. A natural class of solutions are those given by the Lagrangian submanifolds of the phase space when it is endowed with the symplectic structure modified by the electromagnetic field. We have found that the existence of this type of solution leads us directly to the Klein-Gordon equation as a compatibility condition. Therefore, surprisingly, quite natural assumptions on the classical theory involve a quantum condition without any process of limit. This result could be a partial response to the inquiries of Dirac in [1].

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INTRODUCTION

Since the beginning of Quantum Mechanics, physicists have linked the new mechanics to the classical one. This is bound because it is not possible to move forward in the vacuum and by the success of the Newtonian and Maxwellian theories. For example, Schrödinger [6] justifies his famous wave equation by appealing to the classical Hamilton-Jacobi equation. In this paper we will prove that the geometry of the Newton equation (as exposed in [4]) leads naturally to the (generalized) Klein-Gordon equation. More specifically, we will see that the Klein-Gordon equation is a necessary condition for the existence of charge-current fields obeying the Lorentz law, if such fields correspond to Lagrangian submanifolds (of the symplectic structure of the phase space). The result holds for arbitrary pseudo-metrics (on arbitrary dimensional manifolds) included, as particular cases, Riemann and Minkowski metrics.

We want to emphasize the possible relationship of this work with the research of Dirac in [1] (see also [5], pp.190-3, for an interesting and modern treatment and [2, 3] where Dirac himself extends the formalism in order to include more general physical situations). In the above cited paper the author re-examines the classical theory of electrons: he first imposes the constancy of $\|A\|^2$ as a subsidiary condition on the ordinary action principle for the electromagnetic field $F = dA$ (“*the simplest relativistic way of destroying the gauge transformations*”, in his words). Then, a further analysis lead him to consider the vector field associated with the very A as a possible motion of charges. On our part, we consider a natural kind of

solutions of the Lorentz law (Lagrangian submanifolds). It turns out that the motions of charges thus described, consist of certain potentials A which are, necessarily, of constant length. In this way, it seems that we recover the kind of description that Dirac was looking for.

The structure of the paper is as follows. In the first section we exhibit the formulation of Newton's second law as given in [4], where its relationship with the Lagrangian and Hamiltonian formulations becomes transparent. In the second section, we describe the Newton equation when the force is given by the Lorentz law and explain the equivalence with the no-force case at the cost of modifying the underlying symplectic structure. In Section 3, we characterize velocity fields u of flows comprised of particles obeying Newton's law: *field solutions*. Finally, we derive the Klein-Gordon equation for field solutions: in Section 4 for the free case and in Section 5 for the charge-current tensor given by the Maxwell equations.

1. NEWTON'S EQUATION

Let M be an smooth manifold endowed with a non-degenerate metric T_2 whose expression in local coordinates q^μ is

$$T_2 = g_{\mu\nu} dq^\mu dq^\nu, \quad \det(g_{\mu\nu}) \neq 0.$$

The metric T_2 allows us to translate the canonical Liouville 1-form from the cotangent bundle T^*M to the tangent bundle TM . Let us denote by θ this translated form. Locally,

$$\theta = g_{\mu\nu} \dot{q}^\mu dq^\nu,$$

where, as usual, dotted functions mean $\dot{f}(v_x) := v_x(f)$ for all tangent vector $v_x \in T_x M$, $x \in M$ and $f \in C^\infty(M)$. In this way, 2-form $\omega := d\theta$ defines a symplectic structure on TM .

Let us denote by $\pi: TM \rightarrow M$ the natural projection $\pi(v_x) = x$. We will say that an 1-form α defined on the manifold TM is *horizontal* if it kills all of the vertical (with respect to π) tangent vectors. In local coordinates,

$$\alpha = \alpha_\mu dq^\mu, \quad \alpha_\mu = \alpha_\mu(q, \dot{q}).$$

A *second order differential equation* is, by definition, a vector field D on the manifold TM such that

$$\pi_* D_{v_x} = v_x, \quad \forall v_x \in TM.$$

This means that $Df = \dot{f}$ and then, locally,

$$D = \dot{q}^\mu \frac{\partial}{\partial q^\mu} + f^\mu(q, \dot{q}) \frac{\partial}{\partial \dot{q}^\mu},$$

for suitable functions f^μ .

Now, we will establish the Newton's second law " $F = ma$ " in a very convenient fashion.

Definition 1.1. The Newton's equation associated with a horizontal 1-form α ("work form"), is the only second order differential equation D such that

$$(1.1) \quad D \lrcorner \omega + dT + \alpha = 0$$

where T denotes the "kinetic energy" function defined by

$$T(v_x) := \frac{1}{2} T_2(v_x, v_x), \quad \forall v_x \in TM,$$

(so that, locally, $T = (1/2)g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu$).

The reciprocal determination of D and α by means of (1.1) can be viewed in [4], from where we have taken it: each "force" α determines an "acceleration" D and vice versa. This proves the correctness of the above definition. When $\alpha = 0$ (no force is acting on the system), equation (1.1) is the equation of the geodesic curves with respect to the metric T_2 .

2. LORENTZ'S FORCE

An electromagnetic field is defined as a closed 2-form F on M ,

$$F = F_{\mu\nu} dq^\mu \wedge dq^\nu,$$

and a potential for F is an 1-form $A = A_\mu dq^\mu$ such that $F = dA$; that is to say,

$$F_{\mu\nu} = \frac{1}{2} \left(\frac{\partial A_\nu}{\partial q^\mu} - \frac{\partial A_\mu}{\partial q^\nu} \right).$$

Definition 2.1. The Lorentz force associated with F is given by the 1-form α defined by

$$\alpha_{v_x} = v_x \lrcorner F, \quad \forall v_x \in TM.$$

(which depends on the velocities); locally,

$$\alpha = 2F_{\mu\nu}\dot{q}^\mu dq^\nu.$$

The associated Newton's equation (1.1) now is

$$(2.1) \quad D \lrcorner (\omega + F) + dT = 0.$$

Let us introduce the notation

$$\omega_F := \omega + F$$

so that $\omega_F = d(\theta + A)$ if $F = dA$ and equation (2.1) can be rewritten as

$$(2.2) \quad D \lrcorner \omega_F + dT = 0,$$

which is interpreted as follows: D is the Hamiltonian field corresponding to the function T , the kinetic energy, for the symplectic structure (TM, ω_F) .

3. FIELD-SOLUTIONS OF NEWTON'S EQUATION

We will deal now, not with the solution curves of Newton's equation (which is of second order) taken individually, but rather of vector fields whose integral curves are among the former ones.

Let us observe that each parameterized curve on M can be naturally prolonged to TM : it is sufficient to consider the velocity vector at each point.

A vector field $u: M \rightarrow TM$ is said to be an *intermediate integral* of a second order equation $D: TM \rightarrow TTM$ if and only if the prolongation to TM of each integral curve of u is also an integral curve of D .

The prolongation to TM of integral curves of u , completely fills the submanifold $Im(u)$, image of $u: M \rightarrow TM$. In fact, u_*u is the vector field on $Im(u)$ whose integral curves are the prolongation of integral curves of u (we denote by u_* the tangent map associated with map u). As a consequence, the condition for u being an intermediate integral of D is

$$(3.1) \quad D_{u_x} = u_*u_x, \quad \forall x \in M.$$

On the other hand, it follows from the very definition of Liouville 1-form that for an arbitrary vector field u , the restriction of θ to the image of map u is

$$(3.2) \quad u^*\theta = u \lrcorner T_2.$$

(u^* denotes the pull-back induced by u).

The intermediate integrals u for Newton's equation (1.1) will be called *field-solutions*. In this way, and summing up the above considerations, we get the “only if” part of the following characterization.

Proposition 3.1. *A vector field u is a field-solution of Newton's equation (1.1) if and only if*

$$(3.3) \quad u \lrcorner d(u \lrcorner T_2) + dT(u) + u^*\alpha = 0,$$

where $T(u) := u^*T$.

Proof. The “if” part is due to $D_{u_x} - u_*u_x$ being vertical and then the vanishing of $u^*((D_{u_x} - u_*u_x) \lrcorner \omega)$ implies $D_{u_x} - u_*u_x = 0$. \square

4. FIELD-SOLUTIONS FOR THE LORENTZ FORCE

From now on, we will follow and extend a convention from [5]: With each (p, q) tensor field S on M , we will denote S^* the (q, p) tensor that is obtained when the covariant components are converted into contravariant components and vice versa, by means of the metric T_2 . For instance, if u is a tangent vector field on M , u^* will be the 1-form

$$u^* := u \lrcorner T_2,$$

or, in local coordinates,

$$u = u^\mu \frac{\partial}{\partial q^\mu}, \quad u^* = u_\mu dq^\mu,$$

where $u_\mu := g_{\mu\nu}u^\nu$, according to the usual procedure for raising and lowering indexes.

Newton's equation (3.3) for a field u under the Lorentz force induced by $F = dA$ is

$$(4.1) \quad u \lrcorner d(u^* + A) + dT(u) = 0.$$

In such a case, kinetic energy $T(u)$ is a first integral of u .

A particularly important class of field solutions u of (4.1) is given by the set of Lagrangian submanifolds (with respect to ω_F). Explicitly, this is equivalent to consider those tangent fields $u: M \rightarrow TM$ such that, locally,

$$(4.2) \quad u^* + A = df$$

for a local smooth function f defined on M . In this case, we derive $dT(u) = 0$ and then,

Lemma 4.1. *The Lagrangian solutions u of equation (4.1) have constant kinetic energy*

$$(4.3) \quad T(u) = \frac{1}{2}m^2,$$

(m can be real or imaginary, according to the sign of the kinetic energy).

5. KLEIN-GORDON EQUATION FOR CONSERVATIVE GEODESIC FIELDS

As a first case, we will derive the Klein-Gordon equation without coupling; so, no electromagnetic field is present: $F = 0$.

Let δ denote the divergence operator associated with the (pseudo-riemannian) metric T_2 . On vector fields, δ is the standard divergence operator. For an 1-form σ , we have $\delta\sigma = \delta(\sigma^*)$. The Laplacian operator on differential forms is given by

$$\Delta := d\delta + \delta d,$$

which simplifies to $\Delta = \delta d$ on functions (0-forms). As well known, in the case of the Minkowski metric, Δ symbol is substituted by \square and named D'Alembertian operator.

Lemma 5.1. *For each smooth function ϕ on M it holds*

$$\Delta e^{i\phi} = e^{i\phi} (i\Delta\phi - T^2(d\phi, d\phi)).$$

Proof. By definition,

$$\Delta e^{i\phi} = \delta d(e^{i\phi}) = \delta(i e^{i\phi} d\phi) = \delta(i e^{i\phi} (d\phi)^*) = (d\phi)^* (i e^{i\phi}) + i e^{i\phi} \delta(d\phi)^*,$$

where we applied the identity $\delta(\lambda v) = v(\lambda) + \lambda \delta v$ for the function $\lambda = i e^{i\phi}$ and the vector field $v = (d\phi)^*$.

But,

$$(d\phi)^*(ie^{i\phi}) = i^2 e^{i\phi} (d\phi)^*(\phi) = -e^{i\phi} T^2(d\phi, d\phi)$$

and $\delta(d\phi)^* = \delta d\phi = \Delta\phi$, which finishes the proof. \square

Now, let us consider the equation for the geodesic vector fields. This corresponds to take equation (4.1) for $F = 0$,

$$(5.1) \quad u \lrcorner du^* + dT(u) = 0.$$

A class of solutions of (5.1) is given by those vector fields u such that $du^* = 0$. Equivalently, with respect to the symplectic manifold (TM, ω) , we are considering the local Lagrangian submanifolds $u^* = df$ for some function $f \in C^\infty(M)$.

In particular, $T(u) = \frac{1}{2}T_2(u, u) = \frac{1}{2}T_2(df, df) = \frac{1}{2}m^2$.

Moreover, let us assume that field u is conservative:

$$\delta u = 0.$$

Then $\Delta f = 0$ and Lemma 5.1, when applied on $\phi = \frac{1}{\hbar}f$, gives

$$\Delta e^{i\frac{f}{\hbar}} = e^{i\frac{f}{\hbar}} \left(i\frac{1}{\hbar} \Delta f - \frac{1}{\hbar^2} T^2(df, df) \right) = -e^{i\frac{f}{\hbar}} \frac{m^2}{\hbar^2}.$$

So that,

Proposition 5.2. *Conservative geodesic fields $u^* = df$ hold Klein-Gordon equation*

$$\left(\Delta + \frac{m^2}{\hbar^2} \right) \psi = 0,$$

for $\psi := e^{i\frac{f}{\hbar}}$ and $m^2 = T_2(u, u)$.

6. KLEIN-GORDON EQUATION FOR A MAXWELL FIELD

The first Maxwell equation for electromagnetic field $F = dA$ is

$$\delta F = J^*,$$

where J is the so called charge-current vector field. In particular, field J is conservative because $\delta J = \delta^2 F = 0$.

Let us assume that the vector field $u = J$ gives a Lagrangian solution of Newton's equation for the Lorentz force (4.1) so that, locally exists a function f such that

$$J^* + A = u^* + A = df.$$

(This is equivalent to say that $-J^*$ is a vector potential for F as was proposed by Dirac in [1]). In addition, let us choose a potential A with the *Lorentz gauge*

$$\delta A = 0.$$

In this case,

$$\Delta f = \delta u^* + \delta A = \delta J + \delta A = 0.$$

Now, the same computation of the previous section, gives us

$$\Delta e^{i\frac{f}{\hbar}} = e^{i\frac{f}{\hbar}} \left(i\frac{1}{\hbar} \Delta f - \frac{1}{\hbar^2} T^2(df, df) \right) = -\frac{1}{\hbar^2} e^{i\frac{f}{\hbar}} T^2(df, df).$$

On the other hand,

$$m^2 = T^2(u^\star, u^\star) = T^2(-A + df, -A + df) = \|A\|^2 - 2T^2(A, df) + T^2(df, df),$$

so that, taking into account $T^2(A, df) = A^\star(f)$ (= the derivative of f along the vector field A^\star),

$$T^2(df, df) = m^2 - \|A\|^2 + 2A^\star(f).$$

It follows that

$$\Delta e^{i\frac{f}{\hbar}} = -\frac{1}{\hbar^2} e^{i\frac{f}{\hbar}} (m^2 - \|A\|^2 + 2A^\star(f))$$

or

$$\left(\Delta - 2\frac{i}{\hbar} A^\star + \frac{1}{\hbar^2} (m^2 - \|A\|^2) \right) e^{i\frac{f}{\hbar}} = 0$$

where we have applied that $A^\star(e^{i\frac{f}{\hbar}}) = \frac{i}{\hbar} e^{i\frac{f}{\hbar}} A^\star(f)$. Summing up,

Theorem 6.1. *If the Newton equation (4.1) for the Lorentz force given by $F = dA$, $\delta A = 0$, admits a Lagrangian solution $J^\star + A = df$, where $J^\star := \delta F$, then $T_2(J, J)$ is a constant, say m^2 (positive or negative), and the exponential $\psi = e^{i\frac{f}{\hbar}}$ satisfies the Klein-Gordon equation*

$$(6.1) \quad \left(\Delta - 2\frac{i}{\hbar} A^\star + \frac{1}{\hbar^2} (m^2 - \|A\|^2) \right) \psi = 0$$

Because function f is supposed to be real, equation (6.1) for $\psi = e^{if/\hbar}$ implies that $\Delta f = 0$ and $T_2(u, u) = m^2$, for $u^\star := -A + df$. As a consequence, u holds Newton's equation (4.1) and $\delta u = 0$ (and conversely). However, the condition of being $u = \delta F$ can not be assured.

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